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ADRIAN BUTSCHER

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The gluing technique is used to construct hypersurfaces in Euclidean space having approximately constant prescribed mean curvature. These surfaces are perturbations of unions of finitely many spheres of the same radius assembled end-to-end along a line segment. The condition on the existence of these hypersurfaces is the vanishing of the sum of certain integral moments of the spheres with respect to the prescribed mean curvature function.

1. Introduction

In [Butscher and Mazzeo 2008] we have constructed examples of constant mean curvature (CMC) hypersurfaces in a Riemannian manifold M with axial symmetry by gluing together small spheres positioned end-to-end along a geodesic γ. These examples have very large mean curvature $2/r$ and lie within a distance $\mathbb{O}(r)$ of either a segment or a ray of γ; hence we say that these surfaces *condense* to the appropriate subset of γ. Such surfaces cannot exist in Euclidean space, and their existence relies on the fact that the gradient of the ambient scalar curvature of M acts as a "friction term" that permits the usual analytic gluing construction (akin to the classical gluing constructions pioneered by Kapouleas [1990a; 1991]) to be carried out. The purpose of this paper is to show the same techniques used in [Butscher and Mazzeo 2008] can be adapted in a straightforward manner to show that a similar construction is possible in a much simpler yet fairly general context: that of hypersurfaces having *prescribed* near-constant mean curvature in Euclidean space, in a certain sense to be explained forthwith. The essence of the gluing construction carried out herein therefore lies in identifying and appropriately exploiting the analogous friction term appearing in this setting.

Let $F : \mathbb{R}^{n+1} \times T\mathbb{R}^{n+1} \to \mathbb{R}$ be a given, fixed smooth function. For simplicity and to maintain the parallel with the earlier paper, we will assume that F has cylindrical symmetry in the following sense. Endow $\mathbb{R}^{n+1}$ with coordinates $(x^0, x^1, \dots, x^n)$ and let $G \subseteq O(n+1)$ be the set of orthogonal transformations that fix the x^0-axis. Each rotation $R \in G$ acts on $T\mathbb{R}^{n+1}$ via the differential $R_* : T\mathbb{R}^{n+1} \to T\mathbb{R}^{n+1}$. We

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will now demand that $F(R(p), R_* V_p) = F(p, V_p)$ for all $(p, V_p) \in \mathbb{R}^{n+1} \times T\mathbb{R}^{n+1}$. The prescribed mean curvature problem that will be solved in this paper is to find, for every sufficiently small $r \in \mathbb{R}_+$, a G-invariant hypersurface Σ_r which satisfies

$$H[\Sigma_r](p) = 2 + r^2 F(p, N_{\Sigma_r}(p)) \quad \text{for all } p \in \Sigma_r, \tag{1-1}$$

where $H[\Sigma_r]$ is the mean curvature of Σ_r and N_{Σ_r} is the unit normal vector field of Σ_r. Note that we are not "prescribing" mean curvature in the usual sense; i.e., we don't have an *a priori* curvature function in mind that should equal the mean curvature of the hypersurfaces we construct. Instead, we should understand "prescribed mean curvature" to mean that a fixed external quantity (the function F) imposes an extra condition on the geometry of the hypersurface, which must adjust itself in $\mathbb{R}^3$ in order to satisfy this condition. Consequently, we won't know exactly the value of the mean curvature function, but we will know that it is near-constant and that the external condition is satisfied.

The prescribed mean curvature hypersurfaces of this paper will be built by gluing together a finite number K of spheres of radius one (and thus of mean curvature exactly equal to two) whose centers lie on the x^0-axis using small catenoidal necks having the x^0-axis as their axes of symmetry. In order to properly state the Main Theorem, we must make the following definition, which is meant to capture the most important effect of the prescribed mean curvature function F on the surface whose construction is accomplished in this paper.

Definition 1.1. Let S be a compact surface in $\mathbb{R}^{n+1}$. The *F-moment* of S is the quantity

$$\mu_F(S) := \int_S F(x, N_S(x))\, J \,\mathrm{dVol}_S$$

where N_S is the unit normal vector field of S and dVol_S is the induced volume form of S, while $J : S \to \mathbb{R}$ is defined by $J(x) := \langle \partial/\partial x^0, N_S(x)\rangle$ for $x \in S$.

Now let $p_k^0(s) := (s + 2(k-1), 0, \dots, 0)$ and consider the spheres $S_k(s) := \partial B_1(p_k^0(s))$. These spheres are positioned along the x^0-axis in such a way that each $S_k(s)$ makes tangential contact with $S_{k\pm1}(s)$. The following theorem will be proved in this paper.

Main Theorem. *Suppose that there is $s_0 \in \mathbb{R}$ such that*

- *the F-moments of the spheres $S_k(s_0)$ satisfy $\sum_{k=1}^K \mu_F(S_k(s_0)) = 0$, and*
- *the function $s \mapsto \sum_{k=1}^K \mu_F(S_k(s))$ has nonvanishing derivative at $s = s_0$,*

then for all sufficiently small $r > 0$, there is a smooth, embedded hypersurface Σ_r which is a small perturbation of $\bigcup_{k=1}^K S_k(s_0)$ that satisfies the prescribed mean curvature equation (1-1).

It is easy to find a situation in which the conditions of the Main Theorem hold. For example: if $F(\,\cdot\,,\cdot\,)$ is such that $\mu_F(\partial B_1(x^0, x^1, \dots, x^n))$ is negative whenever x^0 is sufficiently negative and positive whenever x^0 is sufficiently positive, the mean value theorem asserts that the function $s \mapsto \sum_{k=1}^{K} \mu_F(S_k(s))$ has a zero. And if also $F(x,\cdot\,)$ is monotone as a function of x^0, this function will have nonzero derivative.

An application of the Main Theorem, and indeed an inspiration for it, is the earlier work by Kapouleas [1990b] on slowly rotating assemblies of water droplets. In this case, the prescribed mean curvature function $F : \mathbb{R}^{n+1} \times T\mathbb{R}^{n+1} \to \mathbb{R}$ takes the form $F(p, N_{\Sigma_r}(p)) := C(\omega)(p^0)^2$ where $p := (p^0, p^1, \dots p^n)$ and $C(\omega)$ depends on the angular velocity ω. The prescribed mean curvature equation now approximates the effect of centrifugal force on the surface Σ_r when ω is small. One of the assemblies of water droplets that Kapouleas constructs is exactly as described in the Main Theorem. (He constructs many other, more complex, and less symmetrical assemblies as well.)

Another application of the Main Theorem is for understanding the possible shapes an electrically charged soap film can adopt in the presence of a weak, axially symmetric electric field. In this case, the equation satisfied by the surface adopted by the soap film is exactly (1-1), where the prescribed mean curvature function $F : \mathbb{R}^{n+1} \times T\mathbb{R}^{n+1} \to \mathbb{R}$ takes the form $F(p, N_{\Sigma_r}(p)) := -C\langle\nabla\phi(p), N_{\Sigma_r}(p)\rangle$ and $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ is the electric potential and C is a constant. We can see why this is so by writing the total energy of the soap film as the sum of a surface area term and a term proportional to the surface integral of ϕ, and then computing the Euler-Lagrange equation for the variation of this energy subject to the constraint that the volume enclosed by the surface remains constant. If we now assume that ϕ is such that the existence conditions of the Main Theorem hold, then the Main Theorem asserts that K spherical, electrically charged soap films connected by small catenoidal necks can be held in equilibrium at special points in space by the electric field.

2. The approximate solution

To construct an approximate solution for the Main Theorem, we use essentially exactly the same procedure as in [Butscher and Mazzeo 2008, §3.1]. This will be outlined here very briefly for the convenience of the reader. The presentation is given for the dimension $n = 2$ for simplicity; everything that follows can be easily adapted to the $(n + 1)$-dimensional setting.

Endow $\mathbb{R}^3$ with coordinates (x^0, x^1, x^2), and let γ be the x^0-axis and $\gamma(t)$ be the arc-length parametrization of the x^0-axis with $\gamma(0) = (0, 0, 0)$. We will construct an approximate solution for the Main Theorem out of K spheres of radius one as follows. Choose a localization parameter $s \in \mathbb{R}$ and small separation parameters $\sigma_1, \dots, \sigma_{K-1} \in \mathbb{R}_+$. Define $s_1 := s$ and $s_k := s + 2(k-1) + \sum_{l=1}^{k-1} \sigma_l$ for $k = 2, \dots, K$ and set $p_k := \gamma(s_k)$ and $p_k^{\pm} := \gamma(s_k \pm 1)$. Define the spheres $S_k := \partial B_1(p_k)$. These spheres will now be joined together according to the following three steps.

Step 1. The first step is to replace each S_k with the surface $\tilde{S}_k$ obtained by taking the normal graph of a specially chosen function G_k over $S_k \setminus [B_{\rho_k}(p_k^+) \cup B_{\rho_k}(p_k^-)]$ where $\rho_k \in (0, 1)$ is a small radius as yet to be determined. The functions we use for this purpose can be defined as follows. Let $\mathscr{L}_{\mathbb{S}^2} := \Delta_{\mathbb{S}^2} + 2$ be the linearized mean curvature operator of the unit sphere, let $\varepsilon_k^{\pm}$ be yet-to-be-determined small scale parameters and let $J_k := \langle \partial/\partial x^0, N_{S_k} \rangle$ be the sole G-invariant function in the kernel of $\mathscr{L}_{\mathbb{S}^2}$ normalized to have unit L^2-norm. Then the functions G_k should satisfy the equations

$$\begin{aligned}
&\mathscr{L}_{\mathbb{S}^2}(G_k) = \varepsilon_k^+ \delta(p_k^+) + \varepsilon_k^- \delta(p_k^-) + A_k J_k && \text{if } k = 2, \dots K-1,\\
&\mathscr{L}_{\mathbb{S}^2}(G_1) = \varepsilon_1^+ \delta(p_1^+) + A_1 J_1 && \text{if } k = 1,\\
&\mathscr{L}_{\mathbb{S}^2}(G_K) = \varepsilon_K^- \delta(p_K^-) + A_K J_K && \text{if } k = K,
\end{aligned}$$

where $\delta(q)$ is the Dirac δ-function centered at q and A_k is chosen to ensure L^2-orthogonality to J_k. (Of course $J_k = x^0|_{S_k}$, the restriction of the x^0 coordinate function to S_k). Furthermore, G_k should be chosen L^2-orthogonal to J_k, normalized to have unit L^2-norm, and to be positive in a neighborhood of p_k^+.

Step 2. Let Ξ be the catenoid, i.e., the unique complete minimal surface of revolution whose axis of symmetry is γ and whose waist lies in the (x^1, x^2)-plane. The next step is to find the truncated and rescaled catenoidal neck of the form $\Xi_k := B_{\rho'_k}(p_k^\flat) \cap [\varepsilon_k \Xi + p_k^\flat + (\delta_k, 0, 0)]$ that fits optimally in the space between $\tilde{S}_k$ and $\tilde{S}_{k+1}$ for $k = 2, \dots, K-1$. Here $\varepsilon_k > 0$ is a small scale parameter and $p_k^\flat$ is a point between p_k^+ and p_{k+1}^- that are determined by the optimal fitting procedure while δ_k is a small vertical displacement parameter that takes Ξ_k away from its optimal location and ρ'_k is a small radius as yet to be determined. The optimal fit is obtained by matching the asymptotic expansions of the functions giving $\tilde{S}_k \cap B_{\rho'_k}(p_k^\flat)$ and $\tilde{S}_{k+1} \cap B_{\rho'_k}(p_k^\flat)$ and Ξ_k as graphs over the translate of the (x^1, x^2)-plane passing through $p_k^\flat$ exactly as in [Butscher and Mazzeo 2008, §3.1]. One particularly important outcome of the matching is that ε_k from the previous step, as well as $\varepsilon_k^{\pm}$ and $p_k^\flat$ are all uniquely determined by σ_k. In fact, an invertible relationship of the form $\sigma_k := \Lambda_k(\varepsilon_k)$ holds, with $\Lambda_k(\varepsilon_k) = \mathbb{O}(\varepsilon_k\, |\log(\varepsilon_k)|)$. Finally, we find that we must choose $\rho_k, \rho'_k = \mathbb{O}(\varepsilon_k^{3/4})$ to ensure the optimal fit between the necks and the perturbed spheres.

Step 3. The final step is to use cut-off functions to smoothly glue the neck Ξ_k into the space between $\tilde{S}_k$ and $\tilde{S}_{k+1}$. In this way we obtain a family of surfaces depending on the σ, δ and s parameters. Denote the neck modified by the cut-off functions by $\tilde{\Xi}_k$. The interpolating region is the annulus $B_{\rho_k'}(p_k^\flat) \setminus B_{\rho_k'/2}(p_k^\flat)$.

Definition 2.1. Let K be given. The approximate solution with parameters $\sigma := \{\sigma_1, \dots, \sigma_{K-1}\}$ and $\delta := \{\delta_1, \dots, \delta_{K-1}\}$ and s is the surface given by

$$\tilde{\Sigma}(\sigma, \delta, s) := \bigcup_{k=1}^{K} \tilde{S}_k \cup \bigcup_{k=1}^{K-1} \tilde{\Xi}_k.$$

3. Solving the projected problem

We now proceed to solve (1-1) up to a finite-dimensional error term by perturbing the approximate solution constructed in the previous section. The required analysis is in most respects identical to or less involved than the analysis found in [Butscher and Mazzeo 2008, SS4–6] and will thus again only be abbreviated here for the sake of the reader. The outcome will be a surface $\Sigma_r^\sharp(\sigma, \delta, s)$ satisfying $H[\Sigma_r^\sharp(\sigma, \delta, s)] - 2 - r^2 F|_{\Sigma_r^\sharp(\sigma,\delta,s)} \in \tilde{\mathcal{W}}$, where $\tilde{\mathcal{W}}$ is a finite-dimensional space of functions that will be defined precisely below. It arises because the linearized mean curvature operator, which governs the solvability of (3-1), possesses a finite-dimensional *approximate kernel* consisting of eigenfunctions corresponding to small eigenvalues. These small eigenvalues make it impossible to implement a convergent algorithm for prescribing the components of the mean curvature of the approximate solution lying in $\tilde{\mathcal{W}}$.

Function spaces. We first define the weighted Hölder spaces in which the analysis will be carried out. These are essentially the same weighted spaces as in [Butscher and Mazzeo 2008, §4], namely the spaces $C^{k,\alpha}_\nu(\tilde{\Sigma}(\sigma, \delta, s))$ consisting of all $C^{k,\alpha}_{\mathrm{loc}}$ functions on $\tilde{\Sigma}(\sigma, \delta, s)$ where the rate of growth in the neck regions of $\tilde{\Sigma}(\sigma, \delta, s)$ is controlled by the parameter ν. Choose some fixed, small $0 < R \ll 1$ and define a weight function $\zeta : \tilde{\Sigma}(\sigma, \delta, s) \to \mathbb{R}$ as

$$\zeta(p) := \begin{cases} \|x\| & \text{for } p = (x^0, x) \in \bar{B}_{R/2}(p_k^\flat) \text{ for some } k, \\ \text{interpolation} & \text{for } p \in \bar{B}_R(p_k^\flat) \setminus B_{R/2}(p_k^\flat) \text{ for some } k, \\ 1 & \text{elsewhere,} \end{cases}$$

where the interpolation is such that ζ is smooth and monotone in the region of interpolation, has appropriately bounded derivatives, and is G-invariant. Now, for any open set $\mathcal{U} \subseteq \tilde{\Sigma}(\sigma, \delta, s)$, define

$$|f|_{C^{k,\alpha}_\nu(\mathcal{U})} := \sum_{i=0}^{k} |\zeta^{i-\nu} \nabla^i f|_{0,\mathcal{U}} + [\zeta^{k+\alpha-\nu} \nabla^k f]_{\alpha,\mathcal{U}},$$

where $|\cdot|_{0,\mathcal{U}}$ is the supremum norm on $\mathcal{U}$ and $[\,\cdot\,]_{\alpha,\mathcal{U}}$ is the α-Hölder coefficient on $\mathcal{U}$. This is the norm that will be used in the $C^{k,\alpha}_\nu(\tilde{\Sigma}(\sigma, \delta, s))$ spaces.

The equation to solve. Let $\mu : C^{2,\alpha}_{\nu}(\tilde\Sigma(\sigma,\delta,s)) \to \mathrm{Emb}(\tilde\Sigma(\sigma,\delta,s),\mathbb{R}^{n+1})$ be the exponential map of $\tilde\Sigma(\sigma,\delta,s)$ in the direction of the unit normal vector field of $\tilde\Sigma(\sigma,\delta,s)$. Hence $\mu_f(\tilde\Sigma(\sigma,\delta,s))$ is the normal deformation of $\tilde\Sigma(\sigma,\delta,s)$ generated by $f \in C^{2,\alpha}_{\nu}(\tilde\Sigma(\sigma,\delta,s))$. The equation

$$H\big[\mu_f(\tilde\Sigma(\sigma,\delta,s))\big] = 2 + r^2\, F \circ \big(\mu_f \times N_{\mu_f(\tilde\Sigma(\sigma,\delta,s))}\big) \tag{3-1}$$

selects $f \in C^{2,\alpha}_{\nu}(\tilde\Sigma(\sigma,\delta,s))$ so that $\mu_f(\tilde\Sigma(\sigma,\delta,s))$ satisfies (1-1). In addition, the function f will be assumed G-invariant. Define the operator

$$\Phi_{r,\sigma,\delta,s} : C^{2,\alpha}_{\nu}(\tilde\Sigma(\sigma,\delta,s)) \to C^{0,\alpha}_{\nu-2}(\tilde\Sigma(\sigma,\delta,s))$$

by

$$\Phi_{r,\sigma,\delta,s}(f) := H\big[\mu_f(\tilde\Sigma(\sigma,\delta,s))\big] - 2 - r^2\,\mathcal{F}(f),$$

where $\mathcal{F}(f) := F \circ (\mu_f \times N_{\mu_f(\tilde\Sigma(\sigma,\delta,s))})$. The linearization of $\Phi_{r,\sigma,\delta,s}$ at zero is given by

$$\begin{aligned}\mathcal{L} &:= \mathrm{D}\Phi_{r,\sigma,\delta,s}(0)\\ &= \Delta + \|B\|^2 + r^2\big(\mathrm{D}_1F(\mu_0, N_{\tilde\Sigma(\sigma,\delta,s)}) \cdot f N_{\tilde\Sigma(\sigma,\delta,s)} - \mathrm{D}_2F(\mu_0, N_{\tilde\Sigma(\sigma,\delta,s)}) \cdot \nabla f\big),\end{aligned}$$

where D_1F and D_2F are the derivatives of F in its first and second slots and $B := B[\tilde\Sigma(\sigma,\delta,s)]$ is the second fundamental form of $\tilde\Sigma(\sigma,\delta,s)$.

The space $\tilde{\mathcal{W}}$ is defined as follows. On the k-th spherical part of $\tilde\Sigma(\sigma,\delta,s)$, the operator $\mathcal{L}$ is a small perturbation of $\mathcal{L}_k := \Delta_{S_k} + 2$ which is the linearized mean curvature operator of the sphere S_k. Let J_k once again be the G-invariant function in its kernel. Now let $\Pi_{\mathrm{ext},k} : \tilde S_k \to S_k \setminus [B_{\rho^k}(p_k^+) \cup B_{\rho^k}(p_k^-)]$ for $k = 1,\dots,K-1$ and also $\Pi_{\mathrm{ext},1} : \tilde S_1 \to S_1 \setminus B_{\rho^k}(p_k^+)$ and $\Pi_{\mathrm{ext},K} : \tilde S_K \to S_K \setminus B_{\rho^k}(p_K^-)$ be the nearest-point projection mappings and define $\tilde J_k := J_k \circ \Pi_{\mathrm{ext},k}$. Finally, let $\chi_{\mathrm{ext},k}$ be a smooth cut-off function supported on $\tilde S_k$ and let η_k be a smooth cut-off function supported on the transition region between the k-th neck and $\tilde S_k$ with the property that the support of $\nabla\eta_k$ and $\nabla\chi_{\mathrm{ext},k}$ do not overlap (this technical assumption is needed in the fine details of the analysis carried out in [Butscher and Mazzeo 2008]).

Definition 3.1. The space $\tilde{\mathcal{W}}$ is defined as

$$\tilde{\mathcal{W}} := \mathrm{span}\{\chi_{\mathrm{ext},k}\tilde J_k : k = 1,\dots,K\} \cup \{\chi_{\mathrm{ext},k}\mathcal{L}_k(\eta_k) : k = 1,\dots,K-1\}.$$

We now prove the following theorem. Let $\varepsilon := \max\{\varepsilon_1,\dots,\varepsilon_{K-1}\}$ and $\delta := \max\{\delta_1,\dots,\delta_{K-1}\}$ and we will assume that $\varepsilon = \mathcal{O}(r^2)$ and $\delta = \mathcal{O}(r)$, which will be justified *a posteriori*.

Theorem 3.2. *If $r > 0$ is sufficiently small, then there exists $f := f_r(\sigma,\delta,s) \in C^{2,\alpha}_{\nu}(\tilde\Sigma(\sigma,\delta,s))$ with $\nu \in (1,2)$ so that*

$$\Phi_{r,\sigma,\delta,s}(f) \in \tilde{\mathcal{W}}. \tag{3-2}$$

The estimate $|f|_{C^{2,\alpha}_\nu} \le Cr^2$ *holds for the function* f, *where the constant C is independent of* r. *Finally, the mapping* $(\sigma, \delta, s) \mapsto f_r(\sigma, \delta, s)$ *is smooth in the sense of Banach spaces.*

Proof. As in [Butscher and Mazzeo 2008], we will use a fixed-point argument to solve the equation $\Phi_{r,\sigma,\delta,s}(f) \in \tilde{\mathcal{W}}$ for a function $f \in C^{2,\alpha}_\nu(\tilde{\Sigma}(\sigma, \delta, s))$ with $\nu \in (1, 2)$. The fixed-point argument follows from three steps: an estimate of the size of $\Phi_{r,s,\sigma,\delta,s}(0)$; the construction of a bounded parametrix $\mathcal{R}$ satisfying $\mathcal{L} \circ \mathcal{R} = \mathrm{id} + \mathcal{E}$ where $\mathcal{E} : C^{0,\alpha}_{\nu-2}(\tilde{\Sigma}(\sigma, \delta, s)) \to \tilde{\mathcal{W}}$; and an estimate of the nonlinear part of the operator $\Phi_{r,\sigma,\delta,s}$. Each of these steps is given in great detail in the paper cited, so we just point out how the analysis there applies to the present situation.

Step 1. We begin with the estimate of $|\Phi_{r,\sigma,\delta,s}(0)|_{C^{0,\alpha}_{\nu-2}}$, the amount that the approximate solution $\tilde{\Sigma}(\sigma, \delta, s)$ deviates from being an actual solution of (3-1). This is done by adapting [Butscher and Mazzeo 2008, Proposition 13]. In fact, by using that proposition's steps 1, 2 and 4 in the estimate of $H[\tilde{\Sigma}(\sigma, \delta, s)] - 2$ in the $C^{0,\alpha}_{\nu-2}$ norm for $\nu \in (1, 2)$, together with a straightforward estimate for the $C^{0,\alpha}_{\nu-2}$ norm of the $r^2\mathcal{F}$ term, we find that

$$\big|\Phi_{r,\sigma,\delta,s}(0)\big|_{C^{0,\alpha}_{\nu-2}} \le C \max\{r^2,\ \varepsilon^{3/2-3\nu/4}, \delta\varepsilon^{1-3\nu/4}\} \le Cr^2$$

for some constant C independent of r.

Step 2. We now find a parametrix

$$\mathcal{R} : C^{0,\alpha}_{\nu-2}(\tilde{\Sigma}(\sigma, \delta, s)) \to C^{2,\alpha}_\nu(\tilde{\Sigma}(\sigma, \delta, s))$$

satisfying $\mathcal{L} \circ \mathcal{R} = \mathrm{id} + \mathcal{E}$, where $\mathcal{E} : C^{0,\alpha}_{\nu-2}(\tilde{\Sigma}(\sigma, \delta, s)) \to \tilde{\mathcal{W}}$. As in [Butscher and Mazzeo 2008, Proposition 15], this is done by first constructing an approximate parametrix by patching together parametrices for the linearized mean curvature operator of each sphere with parametrices for the linearized mean curvature operator of each neck; and then iterating to produce an exact parametrix plus an error term in $\tilde{\mathcal{W}}$ in the limit. The difference here is that the terms coming from the noneuclidean background metric in the result just cited must be replaced by the $r^2\mathcal{F}$ term. The same result holds because this term can easily be shown to satisfy the right estimates. In fact, $\mathcal{R}$ and $\mathcal{E}$ satisfy the estimate $|\mathcal{R}(w)|_{C^{2,\alpha}_\nu} + |\mathcal{E}(w)|_{C^{2,\alpha}_0} \le C|w|_{C^{0,\alpha}_{\nu-2}}$ for all $w \in C^{0,\alpha}_{\nu-2}(\tilde{\Sigma}(\sigma, \delta, s))$, where C is a constant independent of r.

Step 3. We define

$$\mathcal{Q} : C^{2,\alpha}_\nu(\tilde{\Sigma}(\sigma, \delta, s)) \to C^{0,\alpha}_{\nu-2}(\tilde{\Sigma}(\sigma, \delta, s)),$$

the quadratic (and higher) remainder term of the operator $\Phi_{r,\sigma,\delta,s}$, by

$$\mathcal{Q}(f) := \Phi_{r,\sigma,\delta,s}(f) - \Phi_{r,\sigma,\delta,s}(0) - \mathcal{L}(f).$$

The estimates for the $C^{0,\alpha}_\nu$ norm of $\mathcal{Q}$ can be found exactly as in [Butscher and Mazzeo 2008, Proposition 18] with the terms coming from the noneuclidean background metric replaced by the $r^2\mathcal{F}$ term. Then there exists $C_0 > 0$ so that if $f_1, f_2 \in C^{2,\alpha}_\nu(\tilde\Sigma(\sigma,\delta,s))$ for $\nu \in (1,2)$ and satisfying $|f_1|_{C^{2,\alpha}_\nu} + |f_2|_{C^{2,\alpha}_\nu} \le C_0$, then

$$|\mathcal{Q}(f_1) - \mathcal{Q}(f_2)|_{C^{0,\alpha}_{\nu-2}} \le C|f_1 - f_2|_{C^{2,\alpha}_\nu} \max\left\{|f_1|_{C^{2,\alpha}_\nu}, |f_2|_{C^{2,\alpha}_\nu}\right\},$$

where C is a constant independent of r. Once again, this works because the $r^2\mathcal{F}$ term can easily be shown to satisfy the right estimates.

Step 4. We can now solve the CMC equation up to a finite-dimensional error term by implementing a fixed-point argument based on the parametrix constructed in Step 2 as well as the estimates we have computed so far. Let $E := \Phi_{r,\sigma,\delta,s}(0)$ and use the *Ansatz* $f := \mathcal{R}(w - E)$ to convert the equation $\Phi_{r,\sigma,\delta,s}(f) \in \tilde{\mathcal{W}}$ into the fixed point problem $w - \mathcal{N}_{r,\sigma,\delta,s}(w) \in \tilde{\mathcal{W}}$, where

$$\mathcal{N}_{r,\sigma,\delta,s} : C^{0,\alpha}_{\nu-2}(\tilde\Sigma(\sigma,\delta,s)) \to C^{0,\alpha}_{\nu-2}(\tilde\Sigma(\sigma,\delta,s))$$

is defined by

$$\mathcal{N}_{r,\sigma,\delta,s}(w) := -\mathcal{Q} \circ \mathcal{R}(w - E).$$

The estimates established up to now give us

$$|\mathcal{N}_r(w_1) - \mathcal{N}_r(w_2)|_{C^{0,\alpha}_{\nu-2}} \le Cr^2|w_1 - w_2|_{C^{0,\alpha}_{\nu-2}}$$

for w in a ball of radius $\mathcal{O}(r^2)$ about zero in $C^{0,\alpha}_{\nu-2}(\tilde\Sigma(\sigma,\delta,s))$, where C is independent of r. Hence $\mathcal{N}_r$ is a contraction mapping on this ball if r is sufficiently small, and a solution of (3-2) satisfying the desired estimate can be found. The smooth dependence of this solution on the parameters (σ,δ,s) is a consequence of the fixed-point process. □

4. Force balancing arguments and the proof of the Main Theorem

When r is sufficiently small, we have now found a function

$$f_r(\sigma,\delta) \in C^{2,\alpha}_*(\tilde\Sigma(\sigma,\delta,s))$$

for each (σ,δ,s) such that

$$H\big[\mu_{f_r(\sigma,\delta)}(\tilde\Sigma(\sigma,\delta,s))\big] - 2 - r^2\mathcal{F}(f_r(\sigma,\delta,s)) = \mathcal{E}_r(\sigma,\delta,s),$$

where $\mathcal{E}_r(\sigma,\delta,s)$ is an error term belonging to the finite-dimensional space $\tilde{\mathcal{W}}$ depending on the free parameters (σ,δ,s). The corresponding surface that satisfies the prescribed mean curvature condition up to finite-dimensional error is $\Sigma^\sharp_r(\sigma,\delta,s) := \mu_{f_r(\sigma,\delta,s)}(\tilde\Sigma(\sigma,\delta,s))$.

To complete the proof of the Main Theorem, we must show that it is possible to find a value of (σ, δ, s) for which these error terms vanish identically. As in [Butscher and Mazzeo 2008, §7.2], we take cut-off functions $\chi'_{\mathrm{ext},k}$ and $\chi'_{\mathrm{neck},k}$ supported on the k-th spherical region and the k-th neck and transition region, respectively, and consider the *balancing map* $B_r : \mathbb{R}^{2K-1} \to \mathbb{R}^{2K-1}$ defined by

$$B_r(\sigma, \delta, s) := \big(\pi_1(\mathscr{E}_r(\sigma, \delta, s)), \dots, \pi_{2K-1}(\mathscr{E}_r(\sigma, \delta, s))\big), \tag{4-1}$$

where $\pi_{2k+1} : \tilde{\mathscr{W}} \to \mathbb{R}$ and $\pi_{2k} : \tilde{\mathscr{W}} \to \mathbb{R}$ are the L^2-projection operators given by

$$\pi_{2k}(e) := \int_{\tilde{\Sigma}(\sigma,\delta,s)} e \cdot \chi'_{\mathrm{neck},k} \tilde{I}_k, \qquad \pi_{2k+1}(e) := \int_{\tilde{\Sigma}(\sigma,\delta,s)} e \cdot \chi'_{\mathrm{ext},k} \tilde{J}_k .$$

Here $\tilde{I}_k := I_k \circ \Pi_{\mathrm{neck},k}$ where $\Pi_{\mathrm{neck},k}$ is the nearest-point projection mapping of the perturbed k-th neck region onto the unperturbed k-th neck, and I_k is the Jacobi field of the k-th neck coming from translation along the neck axis. This is an odd, bounded function with respect to the center of the neck. Note that B_r is a smooth map between finite-dimensional vector spaces by virtue of the fact that the dependence of the solution $f_r(\sigma, \delta, s)$ on (σ, δ, s) is smooth and the mean curvature operator is a smooth map of the Banach spaces upon which it is defined. The following lemma proves that $\pi(e) = 0$ implies that $e = 0$.

Lemma 4.1. *Choose $e \in \tilde{\mathscr{W}}$ as $e = \sum_{k=1}^{K} a_k \chi_{\mathrm{ext},k} \tilde{J}_k + \sum_{k=1}^{K-1} b_k \mathscr{L}_k(\eta_k)$ for $a_k, b_k \in \mathbb{R}$. Then*

$$\pi_{2k}(e) = C_1 b_k - C_1' \varepsilon_k^{3/2} a_k, \quad \pi_{2k+1}(e) = C_2 a_k,$$

where C_1, C_1', C_2 are positive constants independent of r and (σ, δ, s).

Proof. In the integral

$$\int e \cdot \chi'_{\mathrm{ext},k} \tilde{J}_k = a_k \int \chi_{\mathrm{ext},k} \chi'_{\mathrm{ext},k} \tilde{J}_k^2 + \sum_{\ell=k-1}^{k} b_\ell \int \chi'_{\mathrm{ext},k} \chi_{\mathrm{ext},\ell} \mathscr{L}_\ell(\eta_\ell) \tilde{J}_k,$$

the second two terms can be made to vanish by choosing the supports of χ_{ext}, χ'_{ext} and η_k appropriately. The remaining term has large integral because $\tilde{J}_k = J_k \circ \Pi_{\mathrm{ext},k}$ and J_k has unit L^2 norm as a function of the sphere. In the integral $\int e \cdot \chi'_{\mathrm{neck},k} \tilde{I}_k = \sum_{\ell=k}^{k+1} a_\ell \int \chi_{\mathrm{ext},\ell} \chi'_{\mathrm{neck},k} \tilde{J}_\ell \tilde{I}_k + b_k \int \chi'_{\mathrm{neck},k} \chi_{\mathrm{ext},k} \mathscr{L}_k(\eta_k) \tilde{I}_k$ the first two terms contribute quantities proportional to the volume of the transition regions surrounding the k-th neck where $\chi_{\mathrm{ext},k} \chi'_{\mathrm{neck},k}$ is supported. The remaining term can be made to have large, positive and negative values (depending on the sign of $\tilde{I}_k$) by choosing the supports of χ_{ext}, χ'_{ext} to fall where the quantity $\mathscr{L}_k(\eta_k)$ is largest. □

We must now show that $B_r(\sigma, \delta, s)$ can be controlled by the initial geometry of $\tilde{\Sigma}(\sigma, \delta, s)$, at least to lowest order in r. The calculations are similar to those found in [Butscher and Mazzeo 2008, §7.2] except with the contributions from the ambient background geometry replaced by a contribution from the prescribed mean curvature in the form of the F-moments of the spheres making up $\tilde{\Sigma}(\sigma, \delta, s)$.

The highest-order part of $\mathscr{E}_r(\sigma, \delta, s)$ involves the F-moments of the spherical constituents S_k of $\tilde{\Sigma}(\sigma, \delta, s)$ as follows. Set $\mu_k(\sigma, s) := \mu_F(S_k)$ — this depends on s and $\sigma_1, \dots, \sigma_k$ because the location of the center of S_k is determined by these parameters. Let us continue to assume that $\varepsilon_k = \mathbb{O}(r^2)$ and $\delta_k = \mathbb{O}(r)$ for each k. This will be justified shortly.

Lemma 4.2. *The quantity $\mathscr{E}_r(\sigma, \delta, s)$ satisfies*

$$\pi_{2k}\big(\mathscr{E}_r(\sigma, \delta, s)\big) = C_1 \delta_k \varepsilon_k^{3/2} + \mathbb{O}(r^{2+2\nu}) \tag{4-2}$$

and

$$\pi_{2k+1}\big(\mathscr{E}_r(\sigma, \delta, s)\big) = \begin{cases} C_2 \varepsilon_1 - r^2 \mu_1(\sigma, s) + \mathbb{O}(r^4) & \text{if } k = 0, \\ C_2(\varepsilon_{k+1} - \varepsilon_k) - r^2 \mu_{k+1}(\sigma, s) + \mathbb{O}(r^4) & \text{if } 0 < k < K-1, \\ -C_2 \varepsilon_K - r^2 \mu_K(\sigma, s) + \mathbb{O}(r^4) & \text{if } k = K-1, \end{cases} \tag{4-3}$$

where C_1, C_2 are constants independent of r, σ, δ, s.

Proof. Set $\Sigma_r^\sharp := \Sigma_r^\sharp(\sigma, \delta, s)$ and $\Sigma := \tilde{\Sigma}(\sigma, \delta, s)$ for convenience. Consider first (4-3) with $0 < k < K-1$. By the first variation formula and estimates of the size of the perturbation generating $\Sigma_r^\sharp$ from $\tilde{\Sigma}(\sigma, \delta, s)$, and calculating as in [Butscher and Mazzeo 2008, Proposition 27], we have

$$\begin{aligned} \pi_{2k+1}\big(\mathscr{E}_r(\sigma, \delta, s)\big) &= \int_{\Sigma_r^\sharp} \big(H[\Sigma_r^\sharp] - 2 - r^2 \mathscr{F}(f_r(\sigma, \delta, s))\big) \chi_{\mathrm{ext},k} J_k \\ &= \int_{\partial \Sigma^\sharp \cap \operatorname{supp} \chi_{\mathrm{ext},k}} \Big\langle \frac{\partial}{\partial x^0}, \nu_k \Big\rangle - r^2 \int_{S_k} F(x, N_{S_k}(x)) J_k + \mathbb{O}(r^4) \\ &= C_2\big(\varepsilon_{k+1} - \varepsilon_k\big) - r^2 \mu_k(s, \sigma) + \mathbb{O}(r^4), \end{aligned}$$

where ν_k is the unit normal vector field of $\partial \Sigma^\sharp \cap \operatorname{supp} \chi_{\mathrm{ext},k}$ in $\Sigma^\sharp$.

Now consider (4-2). In the neck we have $H[\tilde{\Sigma}(\sigma, \delta, s)] = 0$. Using similar estimates, we get

$$\begin{aligned} \pi_{2k}\big(\mathscr{E}_r(\sigma, \delta, s)\big) &= \int_{\Sigma_r^\sharp} \big(H[\Sigma_r^\sharp] - 2 - r^2 \mathscr{F}(f_r(\sigma, \delta, s))\big) \chi_{\mathrm{neck},k} I_k \\ &= -2 \int_{\Sigma \cap \operatorname{supp} \chi_{\mathrm{neck},k}} \chi_{\mathrm{neck},k} I_k + \mathbb{O}(r^{2+2\nu}) = C_1 \delta_k \varepsilon_k^{3/2} + O(r^{2+2\nu}), \end{aligned}$$

where δ_k is the displacement parameter of the k-th neck. This is because I_k is an odd function with respect to the neck having $\delta_k = 0$, whereas the integral is being taken over the neck with $\delta_k \neq 0$. Hence the integral $\int_{\Sigma \cap \operatorname{supp} \chi_{\mathrm{neck},k}} \chi_{\mathrm{neck},k} I_k$ picks up the displacement of the k-th neck from its position at $\delta_k = 0$. This same phenomenon arises in [Butscher and Mazzeo 2008, Proposition 27]. □

4.1. ***Proof of the Main Theorem.*** It remains to find a value of the parameters (σ, δ, s) so that $\mathscr{E}_r(\sigma, \delta, s) = 0$. As shown in Lemma 4.1, this is equivalent to find a solution of the equation $B_r(\sigma, \delta, s) = 0$. In what follows, we will continue to assume that $\varepsilon = \mathbb{O}(r^2)$ and $\delta = \mathbb{O}(r)$ and this will be justified shortly. As a consequence of Lemma 4.2, the equations that we must solve are as follows:

$$\begin{aligned}
C_1\delta_1 &= E_1(\sigma, \delta, s),\\
&\ \vdots\\
C_1\delta_{K-1} &= E_{K-1}(\sigma, \delta, s),\\
C_2\varepsilon_1 &= r^2\mu_1(\sigma, s) + E'_1(\sigma, \delta, s),\\
C_2(\varepsilon_2 - \varepsilon_1) &= r^2\mu_2(\sigma, s) + E'_2(\sigma, \delta, s),\\
&\ \vdots\\
C_2(\varepsilon_{K-1} - \varepsilon_{K-2}) &= r^2\mu_{K-1}(\sigma, s) + E'_{K-1}(\sigma, \delta, s),\\
-C_2\varepsilon_{K-1} &= r^2\mu_K(\sigma, s) + E'_K(\sigma, \delta, s),
\end{aligned}$$

where ε_k depends on σ_k in an invertible manner as indicated in Step 2 of the construction of the approximate solution, and E_k, E'_k are error quantities satisfying the bounds $|E_k| = \mathbb{O}(r^{-1+2\nu})$ and $|E'_k| = \mathbb{O}(r^4)$. We can abbreviate these equations by introducing the matrix $M := \left(\begin{smallmatrix} I & 0 \\ 0 & J \end{smallmatrix}\right)$, where I is the $(K-1) \times (K-1)$ identity matrix and J is the $K \times (K-1)$ matrix

$$J := \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{pmatrix}.$$

The equations become

$$M(C_1\delta, C_2\varepsilon)^t = (E, r^2\mu + E')^t, \tag{4-4}$$

where $\delta := (\delta_1, \ldots, \delta_{K-1})$, $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_{K-1})$ and so on for E, E' and μ.

We will solve these equations in two steps as follows. Note first that the matrix M is injective but not surjective, with vectors in the image of M satisfying the relation $(0, e) \cdot M(v, w) = 0$ for all $(v, w) \in \mathbb{R}^{2K-2}$, where $e := (1, 1, \ldots, 1) \in \mathbb{R}^K$. Let $\rho : \mathbb{R}^{2K-1} \to \mathbb{R}^{2K-2}$ be the orthogonal projection onto the image of M. The equation

$$\rho M(\varepsilon,\delta)-\rho(E,r^2\mu+E')=0 \tag{4-5}$$

can now be solved using the implicit function theorem when $r>0$ is sufficiently small if the derivative matrix in (ε,δ) of the mapping on the left hand side of (4-5) above is nonsingular when $r=0$. But this holds because the matrix $\rho M:\mathbb{R}^{2K-2}\to\mathbb{R}^{2K-2}$ is nonsingular and the contribution to the derivative matrix coming from the error term $\rho(E,r^2\mu+E')$ vanishes when $r=0$.

We thus now have a solution $\varepsilon:=\varepsilon_r(s)$ and $\delta_r(s)$ of (4-4) for all sufficiently small r and depending implicitly on the one remaining free parameter s. Moreover, we see that $\varepsilon=\mathbb{O}(r^2)$ and $\delta=\mathbb{O}(r^{-1+2\nu})=\mathbb{O}(r)$ since $\nu\in(1,2)$. It remains to solve (4-5) and we proceed as follows. Once (ε,δ) satisfy (4-5), then (4-4) becomes equivalent to $0=(0,e)\cdot M(\varepsilon,\delta)=r^2e\cdot\mu+e\cdot E'$, or simply

$$\sum_{k=1}^{K}\mu_k(\sigma_r(s),s)+E''(\sigma_r(s),\delta_r(s),s)=0, \tag{4-6}$$

where the error quantity satisfies the estimate $|E''|=\mathbb{O}(r^2)$.

Equation (4-6) may or may not have a solution, depending on the nature of the function $\sum_k\mu_k$, which in turn depends on the specific nature of the prescribed mean curvature function F. However, if the following two conditions are met, then the implicit function theorem guarantees the existence of a solution. First, it must be the case that the equation at $r=0$ has a solution, in other words if the F-moments of the spheres $S_1,\dots,S_K$ satisfy

$$\sum_{k=1}^{K}\mu_F(\partial B_1(p_k^0(s)))=0$$

for some s, where $p_k^0(s):=(s+2(k-1),0,\dots,0)$. Second, if s_0 is the solution of this equation, then it must also be the case that the mapping

$$s\mapsto\sum_{k=1}^{K}\mu_F(\partial B_1(p_k^0(s)))$$

has nonvanishing derivative at $s=s_0$. If these conditions are satisfied, then the implicit function theorem implies that for r sufficiently small, there is a solution $s(r)$ of (4-6). This completes the proof. □

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ADRIAN BUTSCHER
DEPARTMENT OF MATHEMATICS
STANFORD UNIVERSITY
STANFORD, CA 94305
UNITED STATES

butscher@stanford.edu

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