Position-based Elastic Rods

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Ryan Schmidt
Jos Stam
Motivation

• Rods are everywhere
Complexity of Rod’s Behavior

• Rich nonlinear deformation arising from coupling of bending and twisting
Goal

• Fast, robust, simple implementation
• **Qualitatively** good results
• Coupling with other physics models

Position-based Dynamics (PBD)
Elastic Rod Simulations

• All previous works are *not* position-based approach

[Bergou et al. 2008]  [Spillmann et al. 2007]  [Casati et al. 2013]

[Bergou et al. 2010]  [Bartails et al. 2006]  [Pai 2002]
Position-based Dynamics (PBD)

- Elastic rods have yet not been solved using PBD

What is PBD ?: Representation

• Shape is represented in \textit{positions}

\[
\begin{align*}
\mathbf{p}^{t+\Delta t} &= \mathbf{p}^t + \Delta t (\mathbf{v}^t + \Delta t \mathbf{g}) + \Delta \mathbf{p} \\
\mathbf{v}^{t+\Delta t} &= (\mathbf{p}^{t+\Delta t} - \mathbf{p}^t) / \Delta t
\end{align*}
\]
What is PBD?: Elasticity

• Elasticity is handled as constraints

\[ C_1 = |\overline{L}_1 - L_1| = 0 \]
\[ C_2 = |\overline{L}_2 - L_2| = 0 \]
\[ C_3 = |\overline{L}_3 - L_3| = 0 \]
Why Rods are Difficult for PBD?

- **Twist** is difficult to represent with positions

- What is the **constraint**, coupling bend and twist?
Contributions

• **Twist representation** for the position-based rod
  – **Darboux vector** to represent twist and bending
  – **Ghost point** for material frame definition

• **Constraints for twist-bend coupling**
  – **Cossart theory** based formulation

• **Handling of vector type constraint**
  – **Variational formulation**
Outline

Twist representation for PDB

Constraints for twist and bending

Handling vector type constraint

Result

Discussion
Outline

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Material Frame on a Rod

- Adaptive frame: the rod’s tangent = a frame axis
Material Frame Rotation along Rod’s Axis

- Frame is rotating along the black axis
Bending & Twisting as the Frame Rotation

- Coordinate rotation axis is Darboux vector

\[ \Omega_3 = \Omega \cdot d_3 \] (twisting)
\[ \Omega_1 = \Omega \cdot d_1 \] (bending)
\[ \Omega_2 = \Omega \cdot d_2 \]
Ghost Points for Frame Representation

- Ghost point is assigned at the center of edge
Frame Rotation in Discrete Edges

- Constant speed rotation between edges

\[ \Omega = \frac{n\theta}{l} \]
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Continuous Mechanics: Cosserat Theory

• Strain energy is Darboux vector difference

\[ E = \sum_{i=1}^{3} \sum_{j=1}^{3} (\overline{\Omega}_i - \Omega_i) K_{ij} (\overline{\Omega}_j - \Omega_j) \]

rest shape

deformed shape
Constraints for Twist & Bending

• Constraint: Darboux vector do not change

\[
C = \begin{pmatrix}
\Omega_1 \\
\Omega_2 \\
\Omega_3 \\
\end{pmatrix}
- \begin{pmatrix}
\Omega_1 \\
\Omega_2 \\
\Omega_3 \\
\end{pmatrix} = 0
\]
Modification on Discrete Darboux Vector

\[ \Omega = n \theta \approx n2 \tan \frac{\theta}{2} \]

- Good approximation at \( \theta \ll 1 \)
- Prevent flipping at \( \theta \approx \pi \)
Modified Discrete Darboux Vector

• Simple formula without trigonometry

\[ \Omega_i = \left( \frac{2}{l} \right) \frac{d_A^j \cdot d_B^k - d_A^k \cdot d_B^j}{1 + \sum_{n=1}^{3} d_A^n \cdot d_B^n} \]

\( \{i,j,k\} = \{1,2,3\}, \{2,3,1\}, \{3,1,2\} \)
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Vector Type Constraint: Naïve Approaches

\[ \mathbf{C} \in \mathbb{R}^3 \]

- One-by-one approach \( C' = C_i = 0 \)
  - Slow convergence requires many steps 😞

- Energy-based approach \( C' = \|\mathbf{C}\| = 0 \)
  - Slow convergence due to nonlinearity 😞
Gauss’s Principle of Least Constraint

Constrained motion minimize

\[ Z = \sum_{i} m_i (\ddot{p}_i - g) \]

Acceleration change due to constraint

[Carl Friedrich Gauss 1829]
Gauss’s Principle in Discrete Setting

\[ Z = \sum_i m_i |\ddot{p}_i - g|^2 \]

**Continuous representation**

**Constraint update**

\[ \dot{p} = \frac{(v^{t+\Delta t} - v^t)}{\Delta t} = \frac{\Delta p}{\Delta t^2} + g \]

**Verlet integration**

\[ p^{t+\Delta t} = p^t + \Delta t(v^t + \Delta t g) + \Delta p \]

**Position based representation**

\[ Z = \sum_i m_i |\Delta p_i|^2 \]
Variational Interpretation of Constraint Update

\[ \Delta p = \arg \min_{\Delta p} \sum_{s \in S} m_s |\Delta p_s|^2, \text{ where } C(p + \Delta p) = 0 \]

QP problem
linearized constraint

\[ \Delta p_s = -\frac{1}{m_s} \nabla s C^T \left( \sum_{t \in S} \frac{1}{m_t} \nabla t C^T \nabla t C \right)^{-1} C \]

Block Gauss-Seidel

final update formula
Order Dependent Stability

Naïve sequential ordering vs Our bidirectional ordering

Energy Gain Artifact
Coupling with Triangle at End Point
Coupling Rod and Triangle Mesh
Anisotropic Stiffness Adjustment

\[
\begin{align*}
\mathbf{C} &= \begin{pmatrix}
\overline{\Omega}_1 \\
\overline{\Omega}_2 \\
\overline{\Omega}_3
\end{pmatrix} - \begin{pmatrix}
\Omega_1 \\
\Omega_2 \\
\Omega_3
\end{pmatrix} = 0
\end{align*}
\]

\[
\begin{align*}
\Omega_i^{\text{goal}} &= \alpha_i \overline{\Omega}_i + (1 - \alpha_i) \Omega_i \\
\alpha_i &: \text{stiffness}
\end{align*}
\]
Outline

Twist representation for PDB

Constraints for twist and bending

Handling vector type constraint

Result

Discussion
Result: Stability
Result: Large Twist
Result: Wire Mesh
Result: Koosh Ball
Comparison

• Our technique
• Shape matching + twisting force [Rungiratananon et al. 2012]
• Oriented particles [Müller et al. 2011]

• Same discretization and iteration number
• No self-intersection
Our Technique

• Stable coupling twist and bending
Our Technique

• Stable coupling twist and bending
Shape Matching+Twisting Force [Rungiratananon et al. 2012]

- No convergence
Shape Matching+Twisting Force [Rungiratananon et al. 2012]

• Too soft and unstable
Oriented Particles [Müller et al. 2011]

• No coupling between twist and bending
Oriented Particles  [Müller et al. 2011]

• No coupling between twist and bending
## Comparison Summary

<table>
<thead>
<tr>
<th></th>
<th>Stability</th>
<th>Convergence</th>
<th>Twist +Bending</th>
<th>Time per Frame</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Our technique</strong></td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>1.1ms</td>
</tr>
<tr>
<td><strong>Shape Matching +Twisting Force</strong></td>
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<td>❌</td>
<td>✔️</td>
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<td>✔️</td>
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<td>5.0ms</td>
</tr>
</tbody>
</table>
Outline

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Discussion
Application: Hair Design
Future Work

• Preventing flipping completely

• Mathematical proof for stability

• Applications for vector type constraints
  – Continuum mechanics [Muller et al 2014]

\[
F^T F \rightarrow I
\]
• **Summary**
  – First PBD approach for elastic rod simulation

• **Contribution**
  – *Ghost points* for defining frame using positions
  – New constraints formula for twist and bending
  – *Variational Interpretation* for vector constraint enforcement
• Demo is available
  – Autodesk MAYA 2015: Nucleus nHair
    • Bend with “twist tracking” option

Live Demo!

1-3 FPS (include rendering)
• Paper is available
  – Symposium on Computer Animation (SCA 2014)

Position-based Elastic Rods

Nobuyuki Umetani  Ryan Schmidt  Jos Stam
Autodesk Research

Abstract
We present a novel method to simulate complex bending and twisting of elastic rods. Elastic rods are commonly simulated using force based methods, such as the finite element method. These methods are accurate, but do not directly fit into the more efficient position-based dynamics framework, since the definition of material frames are not entirely based on positions. We introduce ghost points, which are additional points defined on edges, so naturally endow continuous material frames on discretized rods. We achieve robustness by a novel discretization of the Cooreman theory. The method supports coupling with a frame, a triangle, and a rigid body at the rod’s end point. Our formulation is highly efficient, capable of simulating hundreds of strands in real-time.

Categories and Subject Descriptors (according to ACM CCS): I.5.8 [Computer Graphics]: Simulation and modeling—Animation

1. Introduction
Position-based dynamics (PBD) has been widely accepted in the field of computer animation due to its efficiency, robustness and simplicity. The goal of the PBD is not to simulate physics as accurately as possible, but rather to sacrifice some quantitative accuracy to generate visually plausible simulation results very quickly. To this end, PBD has broadly been applied in many game engines and visual effects, where speed and controllability is crucial. PBD has primarily been used to simulate various physical phenomena associated with solid and thin-shell (i.e., clothing) deformations [MHK07]. We present a new application of PBD to

Figure 1: A squishy ball hits a wall. Tentacles of the squishy ball was modeled with our position-based elastic rods.

Representation of twist in elastic rods is actively researched, but so far no previous work has successfully
Thanks! Any Questions?

• Acknowledgements
  – Duncan Brinsmead
  – Anonymous reviewers
Variational Interpretation of Constraint Enforcements

\[ \Delta p = \arg \min_{\Delta p} \sum_{s \in S} m_s |\Delta p_s|^2, \quad \text{where} \quad C(p + \Delta p) = 0 \]

\[ \nabla_s \left( \sum_{t \in S} m_t |\Delta p_t|^2 \right) + \lambda^T C = 0 \quad (\forall s \in S) \quad C + \nabla C \Delta p = 0 \]

\[ \Delta p_s = -\frac{1}{m_s} \nabla_s C^T \frac{\lambda}{2} \]

\[ \lambda = 2 \left( \sum_{t \in S} \frac{1}{m_t} \nabla_t C^T \nabla_t C \right)^{-1} C \]

\[ \Delta p_s = -\frac{1}{m_s} \nabla_s C^T \left( \sum_{t \in S} \frac{1}{m_t} \nabla_t C^T \nabla_t C \right)^{-1} C \]
## Removing Bias of Gravity on Ghost Points

<table>
<thead>
<tr>
<th>Method</th>
<th>Gravity</th>
<th>No gravity</th>
<th>Selective gravity</th>
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</thead>
<tbody>
<tr>
<td>Static Hang down</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
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<tr>
<td>Free fall</td>
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<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
</tr>
</tbody>
</table>
Putting it All Together

- Edge length constraint
- Perpendicular bisector constraint
- Distance constraint

Vector-type constraints
- Bending & twisting constraint